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Cayley Inner Functions and Best Approximation

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1. INTRODUCTION

By a *Cayley inner function* we mean a function $\xi(z)$ which is holomorphic and has a positive imaginary part in the open upper half-plane Π_+, whose boundary function $\xi(x + i0)$ is real a.e. on the real line. Equivalently $\xi(z) = i[i - \varphi(z)]/[i + \varphi(z)]$ where $\varphi(z)$ is defined and inner on Π_+ in the usual sense. If $\xi(z)$ is a Cayley inner function we extend its domain of definition to the open lower half-plane Π_- so that $\xi(z^*) = \xi(z)^*$ for all $z \in \Pi_+ \cup \Pi_-$. We also employ the a.e. defined function $\xi(x) = \xi(x + i0) = \xi(x - i0)$ on the real line.

Let $\xi(z)$ be a Cayley inner function, and let (a, b) be a fixed real interval which is not the whole line. If we ignore sets of measure zero, then the real line splits into a disjoint union of a Borel set Δ and its complement Δ^c in such a way that $\xi(x)$ maps Δ onto (a, b) and Δ^c onto the complement of (a, b). We show that conversely if Δ and (a, b) are given, then there is a one-parameter family of Cayley inner functions $\xi(z)$ which have this property with respect to Δ and (a, b). The usefulness of this observation results from the fact that it leads to an evaluation of a large number of definite integrals over Δ in terms of similar integrals over (a, b).

The applications in this paper focus on the circle of ideas surrounding the approximation of x^n by polynomials of lower degree. Pólya (see [6, p. 71]) has shown that if Δ is a closed set with Lebesgue measure $|\Delta|$, then

$$2^{-2n+1} | \Delta |^n \leqslant \min \left\{ \max_{t \in \Delta} \left| t^n + \sum_{j=0}^{n-1} \alpha_j t^j \right| : \alpha_j \text{ complex} \right\} \tag{1}$$

for all $n = 1, 2, 3,\ldots$ and strict inequality holds unless Δ is an interval, in which case there is always equality. When Δ is a disjoint union of r closed

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and bounded intervals, we associate with Δ the $r-1$ points $\lambda_1, \ldots, \lambda_{r-1}$ which satisfy

$$\int_\Delta dt/(t-\lambda) = 0.$$

There is one such point in each of the bounded components of the complement of Δ. We show that

$$2^{-2n+1} \mid \Delta \mid^n = \min \left\{ \max_{t \in \Delta} \left| t^n + \sum_{j=0}^{n-1} \alpha_j t^j \times \sum_{k=1}^{r-1} \sum_{l=0}^{n-1} \beta_{kl}/(t-\lambda_k)^{l+1} \right| : \alpha_j, \beta_{kl} \text{ complex} \right\} \tag{2}$$

for all $n = 1, 2, 3, \ldots$. This result complements Pólya's theorem with a rational approximation scheme (2) which exactly compensates for the error in (1). A similar result is obtained for p-norms.[1]

2. Cayley Inner Functions

The definition of a Cayley inner function has already been given. As in the introduction, we understand that any Cayley inner function $\xi(z)$ is defined on $\Pi_+ \cup \Pi_-$ and there satisfies $\xi(z^*) = \xi(z)^*$. By $\xi(x)$ we mean the a.e. defined function $\xi(x) = \xi(x+i0) = \xi(x-i0)$ on the real line.

Let Δ be a Borel subset of the real line such that neither Δ nor its complement Δ^c has Lebesgue measure zero. Let a, b be extended real numbers such that $a \neq b$ and not both a and b are infinite. We define the *arc* (a, b) to be the usual real point set (a, b) if $a < b$ and $(-\infty, b) \cup (a, \infty)$ if $a > b$. Viewing this on the great circle of real numbers on the Riemann sphere and ignoring end points and the point at infinity, we may regard (a, b), (b, a) as an arbitrary division of the circle into two nontrivial arcs.

2.1 Definition. We say that a Cayley inner function $\xi(z)$ is *adapted* to Δ and (a, b), or somewhat loosely that it *maps* Δ on (a, b), if $\xi(x) \in (a, b)$ for almost all $x \in \Delta$ and $\xi(x) \in (b, a)$ for almost all $x \in \Delta^c$.

Notice that $\xi(z)$ maps Δ on (a, b) if and only if it maps Δ^c on (b, a). It is reasonable to think of the a.e. defined mapping $\xi: \Delta \to (a, b)$ as a "multiplicity" or "rearrangement" function. See the example for the case where Δ is a finite union of intervals, Example 2.3(iv), given later in this section.

[1] *Note added in proof.* Other applications of Cayley inner functions appear in J. D. Chandler, Analysis on unions of intervals, Dissertation, University of Virginia, 1976; and M. Rosenblum and J. Rovnyak, Restrictions of analytic functions, II, *Proc. Amer. Math. Soc.* **51** (1975), 335–343.

We discuss briefly behavior under linear fractional transformations. The general form of a linear fractional transformation which maps Π_+ onto itself is

$$\varphi(z) = (pz + q)/(rz + s)$$

where p, q, r, s are real numbers such that $pq - rs > 0$. It is easy to see that if (a, b) and (c, d) are given arcs, then there is a one parameter family of linear fractional transformations which map Π_+ onto itself and (a, b) onto (c, d). In fact, $z \to pz$, $p > 0$, is the general form of a linear fractional transformation whizh maps Π_+ onto itself and $(0, \infty)$ onto itself. To prove the assertion we map both (a, b) and (c, d) onto $(0, \infty)$ using translations $z \to z + q$, q real, and the inversion $z \to -1/z$, and then we form obvious compositions.

It is immediate from the definitions that if $\xi(z)$ is a Cayley inner function adapted to Δ and (a, b), then $(\varphi \circ \xi)(z) = \varphi(\xi(z))$ is a Cayley inner function adapted to Δ and (c, d) for any linear fractional transformation φ which maps Π_+ onto itself and (a, b) onto (c, d). It turns out that if we fix ξ, then any Cayley inner function η adapted to Δ and (c, d) has the form $\eta = \varphi \circ \xi$ for some such φ. This can be proved using our first result.

2.2 THEOREM. *Let Δ and (a, b) be given. Let $\varphi(z) = (pz + q)/(rz + s)$ be a linear fractional transformation which maps Π_+ onto itself and $(-\infty, 0)$ onto (a, b). Then every Cayley inner function $\xi(z)$ adapted to Δ and (a, b) has a representation in the form*

$$\xi(z) = \frac{p \exp(k + \int_\Delta ((1 + tz)/(t - z))(dt/(1 + t^2))) + q}{r \exp(k + \int_\Delta ((1 + tz)/(t - z))(dt/(1 + t^2))) + s} \tag{3}$$

where k is a real constant, and conversely every function of this form is a Cayley inner function adapted to Δ and (a, b).

Proof. Let $\xi(z)$ be a Cayley inner function adapted to Δ and (a, b). If φ^{in} denotes the linear fractional transformation inverse to φ under composition, then $\varphi^{\text{in}} \circ \xi$ is holomorphic and has positive imaginary part in Π_+. The boundary function of $\varphi^{\text{in}} \circ \xi$ is negative a.e. on Δ and positive a.e. on Δ^c. Therefore (see [1])

$$(\varphi^{\text{in}} \circ \xi)(z) = \exp\left(k + \int_{-\infty}^{+\infty} \frac{1 + tz}{t - z} \frac{f(t)}{1 + t^2}\, dt\right), \qquad z \in \Pi_+,$$

where k is a real constant and

$$f(x) = \lim_{y \searrow 0} \pi^{-1} \arg(\varphi^{\text{in}} \circ \xi)(x + iy) = \chi_\Delta(x)$$

a.e. on $(-\infty, \infty)$. Here χ_Δ denotes the characteristic function of Δ. Thus $\xi(z)$ is given by (3) in Π_+. By symmetry, (3) also holds in Π_-.

The converse statement is proved by checking that

$$\eta(z) = \exp\left(k + \int_{\Delta} \frac{1 + tz}{t - z} \frac{dt}{1 + t^2}\right)$$

is a Cayley inner function adapted to Δ and $(-\infty, 0)$ for any real constant k. Hence $\xi = \varphi \circ \eta$ is a Cayley inner function adapted to Δ and (a, b).

The Nevanlinna representation of a Cayley inner function has the form

$$\xi(z) = \alpha + \rho_0 z + \int_{-\infty}^{+\infty} \frac{1 + tz}{t - z} \frac{d\nu(t)}{1 + t^2}$$

where α and ρ_0 are real constants, $\rho_0 \geqslant 0$, and ν is a singular measure on $(-\infty, \infty)$ such that $\int_{-\infty}^{+\infty} (1 + t^2)^{-1}\, d\nu(t) < \infty$. Conversely, every nonconstant function having this form is a Cayley inner function. These assertions follow immediately from the general theory of functions having positive imaginary part in Π_+ (see [1]).

2.3 Examples. (i) The function $\xi(z) = z$ is a Cayley inner function which maps any arc $\Delta = (a, b)$ on itself. Using Theorem 2.2 and the relations

$$\exp\left(\int_0^{\infty} \frac{1 + tz}{t - z} \frac{dt}{1 + t^2}\right) = \frac{-1}{z}$$

$$\exp\left(\int_a^b \frac{dt}{t - z}\right) = \frac{b - z}{a - z}$$

$-\infty < a < b < \infty$, $z \in \Pi_+ \cup \Pi_-$, we see that the most general Cayley inner function which maps

(a) $\Delta = (-1, 1)$ on itself is $\xi(z) = (z - \sigma)/(1 - \sigma z)$, $-1 < \sigma < 1$,
(b) $\Delta = (0, \infty)$ on itself is $\xi(z) = \tau z$, $0 < \tau < \infty$,
(c) $\Delta = (0, \infty)$ on $(-1, 1)$ is $\xi(z) = (z - \rho)/(z + \rho)$, $0 < \rho < \infty$,
(d) $\Delta = (1, -1)$ on $(-1, 1)$ is $(1 - \sigma z)/(\sigma - z)$, $-1 < \sigma < 1$.

(ii) In case Δ is a finite union of arcs, and only in this case, any Cayley inner function $\xi(z)$ adapted to Δ and some arc (a, b) is a rational function.

(iii) Let both Δ and (a, b) be bounded, so necessarily $-\infty < a < b < \infty$. It follows from Theorem 2.2 that there is a unique Cayley inner function $\xi(z)$ adapted to Δ and (a, b) which has a pole at infinity. It is given by

$$\xi(z) = \frac{a \exp(\int_{\Delta} dt/(t - z)) - b}{\exp(\int_{\Delta} dt/(t - z)) - 1}. \tag{4}$$

The Nevanlinna representation in this case has the form

$$\xi(z) = \alpha + \frac{b-a}{|\Delta|} z + \int_K \frac{d\nu(t)}{t-z}$$

where α is a real constant, $|\Delta|$ is the Lebesgue measure of Δ, K is the smallest closed interval which essentially contains Δ, and ν is a finite singular measure supported on K. This is proved by showing first that $\xi(z)$ is analytic in the complex plane slit along K, and then that

$$\lim_{y\to\infty} \xi(iy)/(iy) = (b-a)/|\Delta|. \tag{5}$$

(iv) We examine in more detail the case where (ii) and (iii) overlap, i.e., the case where $\Delta = \bigcup_1^r (a_j, b_j)$ where $-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r < \infty$ and (a, b) is a bounded open interval. The Cayley inner function $\xi(z)$ in this case will always be chosen by (4), or equivalently

$$\xi(z) = \frac{b - a\Pi_1^r(b_j - z)/(a_j - z)}{1 - \Pi_1^r(b_j - z)/(a_j - z)}. \tag{6}$$

It is easy to see that in each interval (a_j, b_j) in Δ, $\xi(x)$ increases from a to b. In each interval (b_j, a_{j+1}), $\Pi_1^r[(b_j - x)/(a_j - x)]$ increases from 0 to ∞, and therefore $\xi(z)$ has exactly one pole λ_j in (b_j, a_{j+1}). Clearly $\xi(z)$ has a pole at ∞ and exactly r poles counting multiplicity in the extended complex plane. The Nevanlinna representation coincides with the partial fractions decomposition in this case, and it has the form

$$\xi(z) = \alpha + \frac{b-a}{|\Delta|} z + \sum_{j=1}^{r-1} \frac{\rho_j}{\lambda_j - z} \tag{7}$$

where α is a real constant, $|\Delta| = \int_\Delta dt$, $\lambda_1, \ldots, \lambda_{r-1}$ are the roots of

$$\int_\Delta dt/(t-\lambda) = 0, \tag{8}$$

and

$$\rho_j = (b-a)\Big/\int_\Delta dt/(t-\lambda_j)^2, \qquad j = 1, \ldots, r-1. \tag{9}$$

We justify the last two assertions. By (4), $\lambda_1, \ldots, \lambda_{r-1}$ are the roots of

$$\exp\left(\int_\Delta dt/(t-\lambda)\right) = 1$$

which is equivalent to (8). We obtain (9) by computing the residues of the poles in (7):

$$\begin{aligned}\rho_j &= \lim_{x\to\lambda_j} (\lambda_j - x)\left[a \exp\left(\int_\Delta dt/(t-x)\right) - b\right]\Big/\left[\exp\left(\int_\Delta dt/(t-x)\right) - 1\right] \\ &= \lim_{x\to\lambda_j} (\lambda_j - x)(b-a)\Big/\left[1 - \exp\left(\int_\Delta dt/(t-x)\right)\right] \\ &= -(b-a)\left[(d/dx)\exp\left(\int_\Delta dt/(t-x)\right)\right]_{x=\lambda_j} \\ &= (b-a)\Big/\int_\Delta dt/(t-\lambda_j)^2\end{aligned}$$

for all $j = 1,\ldots, r-1$. The applications in Section 4 depend on a technical lemma which we state as

2.4 LEMMA. *Let Δ, (a, b), and $\xi(z)$ be given as in Example* 2.3(iv). *Let $\mathscr{K}_0$ be the linear span of functions on Δ which have the form $k_0(t) = [\xi(t) - \xi(u)]/(t-u)$ where $u \in \Pi_+ \cup \Pi_-$. For each $n = 1, 2, 3,\ldots$ let $\mathscr{K}_n$ be the linear span of the functions on Δ of the form*

$$k_0, k_1\xi, \ldots, k_n\xi^n, \qquad \textit{where} \qquad k_0, k_1, \ldots, k_n \in \mathscr{K}_0 .$$

Then for each $n = 0, 1, 2,\ldots, \mathscr{K}_n$ is $(n+1)$ r-dimensional, and it coincides with the linear span of functions on Δ of the form

$$1, t, \ldots, t^n \quad \text{and} \quad \frac{1}{t-\lambda_j}, \frac{1}{(t-\lambda_j)^2}, \ldots, \frac{1}{(t-\lambda_j)^{n+1}}, \qquad j = 1,\ldots, r-1. \tag{10}$$

Proof. First take $n = 0$. For every $u \in \Pi_+ \cup \Pi_-$,

$$\frac{\xi(t) - \xi(u)}{t-u} = \frac{b-a}{|\Delta|} + \sum_{j=1}^{r-1} \frac{\rho_j}{(\lambda_j - u)(\lambda_j - t)}$$

by (7). If we write this equation for $n+1$ distinct values of u, then it is not hard to see that the resulting system of equations can be solved for 1, $1/(t-\lambda_1),\ldots, 1/(t-\lambda_{r-1})$ as linear combinations of functions $[\xi(t) - \xi(u)]/(t-u)$, u complex. In fact, the system is linear and the coefficient matrix can be reduced to the identity by obvious elementary row and column operations. The assertion follows for the case $n = 0$.

The general case is proved by induction. For any $n = 0, 1, 2,\ldots,$ let $\mathscr{M}_n$ denote the span of the functions (10), and let $\mathscr{L}_n$ be the span of the functions $\xi(t)^n, \xi(t)^n/(t-\lambda_1),\ldots, \xi(t)^n/(t-\lambda_{r-1})$. Then $\dim \mathscr{M}_n = (n+1)r$ and

$\dim \mathscr{L}_n = r$ for all $n = 0, 1, 2, \ldots$. We have shown that $\mathscr{K}_0 = \mathscr{M}_0$. Suppose that $\mathscr{K}_n = \mathscr{M}_n$ for some nonnegative integer n. Then $\mathscr{M}_n = \mathscr{K}_n \subseteq \mathscr{K}_{n+1}$, $\mathscr{L}_{n+1} \subseteq \mathscr{K}_{n+1}$, and $\mathscr{M}_n \cap \mathscr{L}_{n+1} = (0)$. Hence $\dim \mathscr{K}_{n+1} \geqslant (n+1)r + r = (n+2)r$. But $\mathscr{K}_{n-1} \subseteq \mathscr{M}_{n+1}$ where $\dim \mathscr{M}_{n+1} = (n+2)r$. Therefore $\mathscr{K}_{n+1} = \mathscr{M}_{n+1}$, and the result follows by induction.

3. Substitution Theory for Definite Integrals

The main result of this section is Theorem 3.3. The proof is based on some disk results which we develop first.

Let $B(\zeta)$ be an inner function on the open unit disk D_+. Let $\{\zeta_j\}_{j\geqslant 1}$ be the nonzero zeros of $B(\zeta)$ in D_+. We extend $B(\zeta)$ to the complement D_- of $\bar{D}_+$, excluding the points $\{1/\zeta_j{}^*\}_{j\geqslant 1}$, by requiring that $B(\zeta)\, B(1/\zeta^*)^* = 1$ for $\zeta \in D_- \backslash \{1/\zeta_j{}^*\}_{j\geqslant 1}$. It is known (see [4, p. 350]) that the boundary function

$$B(e^{i\theta}) = \lim_{r\to 1} B(re^{i\theta})$$

maps no Borel subset of the unit circle Γ having positive (linear Lebesgue) measure into a Borel subset of Γ of measure zero. This fact is used repeatedly in the following form. If f, g are a.e. defined functions on Γ, then $f \circ B$, $g \circ B$ are a.e. defined functions on Γ, and if $f = g$ a.e. on Γ, then $f \circ B = g \circ B$ a.e. on Γ. In other words "composition with B" is a meaningful operation in the class of a.e. defined functions on Γ.

3.1 Lemma. *If α, β are two points in $D_+ \cup (D_- \backslash \{1/\zeta_j{}^*\}_{j\geqslant 1})$, then for all $m, n = 0, 1, 2, \ldots$,*

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{1 - B(e^{i\theta})\, B(\alpha)^*}{1 - \alpha^* e^{i\theta}}\, B(e^{i\theta})^n\, \frac{1 - B(e^{i\theta})^*\, B(\beta)}{1 - \beta e^{-i\theta}}\, B(e^{i\theta})^{*m}\, d\theta$$
$$= \delta_{mn} \frac{1 - B(\beta)\, B(\alpha)^*}{1 - \beta\alpha^*} \tag{11}$$

where δ_{mn} denotes the Kronecker delta. We define $[1 - B(\beta)\, B(\alpha)^*]/(1 - \beta\alpha^*)$ by continuity when $\beta\alpha^* = 1$.

Proof. This follows, for instance, from [2, Problems 22. 89]. We sketch a direct proof for completeness.

It may be assumed that $n \geqslant m$. The identity

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{h(e^{i\theta})}{1 - \gamma e^{-i\theta}}\, d\theta = h(\gamma) \tag{12}$$

holds for all h in the Hardy space H^1 and all $\gamma \in D_+$. If $\alpha, \beta \in D_+$, then we write the integral in (11) in expanded form as the sum of four integrals. Three of these integrals may be evaluated directly by (12); the fourth one is also evaluated using (12), except that when $m = n$ it is first necessary to conjugate. The identity follows in this case.

If $\alpha, \beta \in D_- \backslash \{1/\zeta_j^*\}_{j \geqslant 1}$, then we obtain (11) by a similar calculation using

$$(1 - \alpha^* e^{i\theta})(1 - \beta e^{-i\theta}) = \beta\alpha^*(1 - e^{-i\theta}/\alpha^*)(1 - e^{i\theta}/\beta).$$

Next let $\alpha \in D_- \backslash \{1/\zeta_j^*\}_{j \geqslant 1}$, $\beta \in D_+$, and suppose $\beta\alpha^* \neq 1$. Then

$$\begin{aligned}&(1 - \alpha^* e^{i\theta})^{-1}(1 - \beta e^{-i\theta})^{-1} \\ &\qquad = (1 - \beta\alpha^*)^{-1}[(1 - \beta e^{-i\theta})^{-1} - (1 - e^{-i\theta}/\alpha^*)^{-1}]\end{aligned}$$

and the result follows in the same way. The restriction $\beta\alpha^* \neq 1$ is removed by a continuity argument.

The remaining case $\alpha \in D_+$, $\beta \in D_- \backslash \{1/\zeta_j^*\}_{j \geqslant 1}$ is handled similarly.

3.2 THEOREM. *Let α, β be two points in $D_+ \cup (D_- \backslash \{1/\zeta_j^*\}_{j \geqslant 1})$. Then for any $g \in L^1(-\pi, \pi)$, $g \circ B \in L^1(-\pi, \pi)$ and*

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{1 - B(e^{i\theta})\, B(\alpha)^*}{1 - \alpha^* e^{i\theta}} \, \frac{1 - B(e^{i\theta})^*\, B(\beta)}{1 - \beta e^{-i\theta}} \, g(B(e^{i\theta}))\, d\theta \\ &\qquad = \frac{1 - B(\beta)\, B(\alpha)^*}{1 - \beta\alpha^*} \cdot \frac{1}{2\pi}\int_{-\pi}^{+\pi} g(e^{i\theta})\, d\theta \qquad (13)\end{aligned}$$

where the value of $[1 - B(\beta)\, B(\alpha)^*]/(1 - \beta\alpha^*)$ *is determined by continuity if* $\beta\alpha^* = 1$.

Proof. By Lemma 3.1, (13) is true if $g(e^{i\theta}) = f(e^{i\theta})^* f(e^{i\theta})$ where f is a polynomial in $e^{i\theta}$. By a theorem of Ryff [4, Theorem 1], $g \to g \circ B$ is a bounded operator on H^2, and therefore (13) holds if $g(e^{i\theta}) = f(e^{i\theta})^* f(e^{i\theta})$ where $f \in H^2$. The proof is completed by showing that every $g \in L^1(-\pi, \pi)$ is a linear combination of such functions. To see this, first write g as a linear combination of integrable functions h such that $h \geqslant 1$ a.e. on $(-\pi, \pi)$. Then use the standard construction of outer functions to show that such h has the form $h(e^{i\theta}) = f(e^{i\theta})^* f(e^{i\theta})$ where $f \in H^2$. The result follows.

We obtain the main result of this section by transformations which take D_+, D_- to Π_+, Π_- respectively and the inner function $B(\zeta)$ to a Cayley inner function $\xi(z)$.

3.3 THEOREM. *Let $\xi(z)$ be a Cayley inner function adapted to Δ and (a, b), where Δ is any Borel set such that neither Δ nor Δ^c has Lebesgue measure zero and (a, b) is any arc.*

(i) *If* $f \in L^1(-\infty, \infty)$, *then*

$$\int_{-\infty}^{+\infty} |f(\xi(t))| \frac{\xi(t)^2 + 1}{t^2 + 1} dt < \infty \tag{14}$$

and

$$\int_{-\infty}^{+\infty} \frac{\xi(t) - \xi(u)^*}{t - u^*} \frac{\xi(t) - \xi(v)}{t - v} f(\xi(t)) \, dt = \frac{\xi(v) - \xi(u)^*}{v - u^*} \int_{-\infty}^{+\infty} f(t) \, dt \tag{15}$$

for all $u, v \in \Pi_+ \cup \Pi_-$.

(ii) *If* $f \in L^1(a, b)$, *then*

$$\int_{\Delta} |f(\xi(t))| \frac{\xi(t)^2 + 1}{t^2 + 1} dt < \infty \tag{16}$$

and

$$\int_{\Delta} \frac{\xi(t) - \xi(u)^*}{t - u^*} \frac{\xi(t) - \xi(v)}{t - v} f(\xi(t)) \, dt = \frac{\xi(v) - \xi(u)^*}{v - u^*} \int_{(a,b)} f(t) \, dt \tag{17}$$

for all $u, v \in \Pi_+ \cup \Pi_-$.

Proof. We first note that by the remarks preceding the statement of Lemma 3.1 and the transformations used in this proof, "composition with ξ" is a meaningful operation in the class of a.e. defined functions on the real line.

The mapping $\zeta \to z = i(1 + \zeta)/(1 - \zeta)$ carries D_+ onto Π_+, D_- onto Π_-, and the points $\{1/\zeta_j^*\}_{j\geq 1}$ onto certain points $\{z_j^*\}_{j\geq 1}$. Here we understand that

$$B(\zeta) = [\xi(z) - i]/[\xi(z) + i]$$

whenever $\zeta \in D_+ \cup (D_- \backslash \{1/\zeta_j^*\}_{j\geq 1})$ and $z \in \Pi_+ \cup (\Pi_- \backslash \{z_j^*\}_{j\geq 1})$ are corresponding points.

Now let $f \in L^1(-\infty, \infty)$. Define $g(e^{i\theta})$ on the unit circle Γ by

$$f(t) = \frac{2}{1 + t^2} g\left(\frac{t - i}{t + i}\right).$$

Then $g \in L^1(-\pi, \pi)$, so by Theorem 3.2, $g \circ B \in L^1(-\pi, \pi)$ and

$$\int_{-\infty}^{+\infty} |f(\xi(t))| \frac{\xi(t)^2 + 1}{t^2 + 1} dt = \int_{-\pi}^{+\pi} |g(B(e^{i\theta})| \, d\theta < \infty.$$

This proves (14). Similarly (15) follows from (13) in case

$$u = i(1 + \alpha)/(1 - \alpha), \qquad v = i(1 + \beta)/(1 - \beta)$$

where α, β are points in $D_+ \cup (D_- \backslash \{1/\zeta_j^*\}_{j\geqslant 1})$. By continuity, (15) holds for the exceptional values of u and v also.

The proof of (i) is complete. We obtain (ii) by specializing (i) to functions which vanish off (a, b).

The applications require some special formulas which are valid under the assumptions in Example 2.3(iv). We state these somewhat more generally under the assumptions of Example 2.3(iii), but we do not seek maximum generality for the formulas.

3.4 THEOREM. *Let Δ, (a, b), and $\xi(z)$ be chosen as in Example* 2.3(iii). *Let $\mathscr{K}_0$ be the smallest weak* closed subspace of $L^\infty(\Delta)$ which contains all functions $[\xi(t) - \xi(u)]/(t - u)$ where $u \in \Pi_+ \cup \Pi_-$. If $f \in L^1(a, b)$, then $f \circ \xi \in L^1(\Delta)$ and*

$$\int_\Delta k(t) f(\xi(t))\, dt = \int_\Delta k(t)\, dt \cdot \frac{1}{b-a} \int_a^b f(t)\, dt \tag{18}$$

$$\int_\Delta k_1(t)\, k_2(t) f(\xi(t))\, dt = \int_\Delta k_1(t)\, k_2(t)\, dt \cdot \frac{1}{b-a} \int_a^b f(t)\, dt \tag{19}$$

for all $k, k_1, k_2 \in \mathscr{K}_0$. We also have

$$\int_\Delta f(\xi(t))\, dt = \frac{|\Delta|}{b-a} \int_a^b f(t)\, dt. \tag{20}$$

Proof. We obtain $f \circ \xi \in L^1(\Delta)$ by (16) and the fact that $\xi \in L^\infty(\Delta)$. By the special case of (17) with $f \equiv 1$,

$$\frac{1}{b-a} \int_\Delta \frac{\xi(t) - \xi(u)^*}{t - u^*} \frac{\xi(t) - \xi(v)}{t - v}\, dt = \frac{\xi(v) - \xi(u)^*}{v - u^*}$$

for all $u, v \in \Pi_+ \cup \Pi_-$. Using this formula we readily deduce (19) for the case where k_1, k_2 are linear combinations of functions $[\xi(t) - \xi(u)]/(t - u)$, $u \in \Pi_+ \cup \Pi_-$. The general case of (19) follows by approximation. By (5)

$$\lim_{y\to\infty} \frac{|\Delta|}{b-a} \frac{\xi(t) - \xi(iy)}{t - iy} = 1$$

a.e. on Δ, and so $1 \in \mathscr{K}_0$. Thus (18) and (20) are special cases of (19).

4. RATIONAL APPROXIMATION ON A FINITE UNION OF INTERVALS

Let X be a complex Banach space, and let Q be a linear subspace of X. If $f \in X$, then an element q_0 of Q is said to be a *best approximation* to f from Q if

$$\|f - q_0\| = \min\{\|f - q\|: q \in Q\}.$$

In this case we say that $f - q_0$ is *orthogonal* to Q, and we write $f - q_0 \perp Q$. If $f \in X$, then $f \perp Q$ if and only if $\|f\| \leqslant \|f + q\|$ for all $q \in Q$. There is a well-known condition for orthogonality in case X is a Lebesgue space, say $X = L^p(\mu)$, $1 \leqslant p < \infty$, where μ is a σ-finite measure.

4.1 *Orthogonality condition.* Let $f \in L^p(\mu)$, $1 \leqslant p < \infty$, and in case $p = 1$ assume that $\{x: f(x) = 0\}$ has measure zero. Let Q be a linear subspace of $L^p(\mu)$. Then $f \perp Q$ if and only if

$$\int |f|^{p-1}(\operatorname{sgn} f)^* \, q \, d\mu = 0, \qquad q \in Q$$

where $\operatorname{sgn} z = z/|z|$ for $z \neq 0$ and $\operatorname{sgn} 0 = 0$.

For a proof see [5, pp. 55–56].

Throughout the rest of the paper we take $\Delta = \bigcup_1^r (a_j, b_j)$ where $-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r < \infty$. Let (a, b) be any real interval whose length $b - a = |\Delta|$ is the sum of the lengths of the intervals in Δ. Let $\xi(z)$ be given by (6), and let $\lambda_1, \ldots, \lambda_{r-1}$ be the points determined by (8). By $\mathscr{K}_0$ we mean the space of functions defined in Lemma 2.4, which is the same as the space described in Theorem 3.4.

4.2 Theorem. *Assume $1 \leqslant p < \infty$. Let $Q(a, b)$ be a linear subspace of $L^p(a, b)$, and let $Q(\Delta)$ be the linear subspace of $L^p(\Delta)$ spanned by all functions of the form $k(x)\, q(\xi(x))$ where $k \in \mathscr{K}_0$ and $q \in Q(a, b)$. Let $f \in L^p(a, b)$, and let q_0 be a best approximation to f from $Q(a, b)$. In case $p = 1$, assume further that $\{x: f(x) - q_0(x) = 0\}$ has measure zero. Then in $L^p(\Delta)$, $q_0 \circ \xi$ is a best approximation to $f \circ \xi$ from $Q(\Delta)$, and*

$$\|f \circ \xi - q_0 \circ \xi\|_{L^p(\Delta)} = \|f - q_0\|_{L^p(a,b)} \,. \tag{21}$$

Proof. By Theorem 3.4 and the necessity of condition 4.1,

$$\begin{aligned}
&\int_\Delta |f(\xi(t)) - q_0(\xi(t))|^{p-1} \{\operatorname{sgn}[f(\xi(t)) - q_0(\xi(t))]\}^* \, k(t)\, q(\xi(t))\, dt \\
&\quad = \int_\Delta k(t)\, dt \cdot \frac{1}{b-a} \int_a^b |f(t) - q_0(t)|^{p-1} \{\operatorname{sgn}[f(t) - q_0(t)]\}^* \, q(t)\, dt \\
&\quad = 0
\end{aligned}$$

for all $k \in \mathscr{K}_0$ and $q \in Q(a, b)$. By the sufficiency of condition 4.1, $q_0 \circ \xi$ is a best approximation to $f \circ \xi$ from $Q(\Delta)$. By Theorem 3.4 and the assumption that $b - a = |\Delta|$,

$$\int_\Delta |f(\xi(t)) - q_0(\xi(t))|^p \, dt = \int_a^b |f(t) - q_0(t)|^p \, dt.$$

and the proof is complete.

For every number p, $1 \leqslant p \leqslant \infty$, and every $n = 0, 1, 2, \ldots$, there is a unique polynomial $P_n^{(p)}(t)$ in the class $\mathscr{P}_n$ of all polynomials of the form

$$P(t) = t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_0, \qquad \alpha_0, \ldots, \alpha_{n-1} \text{ complex},$$

for which the minimum

$$m_n^{(p)}(a, b) = \min\{\| P \|_{L^p(a,b)} : P \in \mathscr{P}_n\}$$

is attained. When $p = 1, \infty$,

$$\begin{aligned} m_n^{(1)}(a, b) &= (b-a)^{n+1}/2^{2n}, \\ P_n^{(1)}(t) &= 2^{-2n}(b-a)^n\, U_n((2t-a-b)/(b-a)) \\ m_n^{(\infty)}(a, b) &= (b-a)^n/2^{2n-1}, \\ P_n^{(\infty)}(t) &= 2^{-2n+1}(b-a)^n\, T_n((2t-a-b)/(b-a)) \end{aligned}$$

for all $n = 0, 1, 2, \ldots$, where $T_n(x)$, $U_n(x)$ are the Chebychev polynomials of the first and second kind, respectively (see [6]).

We construct a rational approximation scheme which has exactly the same minimum deviation from t^n over Δ as polynomial approximation over an interval of length $|\Delta|$.

4.3 THEOREM. *For every number p, $1 \leqslant p \leqslant \infty$, and every $n = 1, 2, 3, \ldots$,*

$$m_n^{(p)}(0, |\Delta|) = \min \left\{ \left\| t^n + \sum_{j=0}^{n-1} \alpha_j t^j + \sum_{k=1}^{r-1} \sum_{l=0}^{n-1} \beta_{kl}/(t-\lambda_k)^{l+1} \right\|_{L^p(\Delta)} : \alpha_j, \beta_{kl} \text{ complex} \right\}, \tag{22}$$

and the minimum is attained for the function $P_n^{(p)}(\xi(t))$.

Proof. Let $Q(a, b)$ be the subspace of $L^p(a, b)$ of polynomials of degree at most $n-1$, and let $Q(\Delta)$ be the subspace of $L^p(\Delta)$ defined in Theorem 4.2. By Lemma 2.4, $Q(\Delta)$ is the set of functions on Δ of the form

$$q(t) = \sum_{j=1}^{n-1} \alpha_j t^j + \sum_{k=1}^{r-1} \sum_{l=0}^{n-1} \beta_{kl}/(t-\lambda_k)^{l+1}$$

where α_j, β_{kl} are complex. Let $f(t) = t^n$.

Suppose first that $1 \leqslant p < \infty$. If q_0 is the best approximation to f from $Q(a, b)$, then $f - q_0 = P_n^{(p)}$. By Theorem 4.2, $q_0 \circ \xi$ is a best approximation to $f \circ \xi$ from $Q(\Delta)$ and (21) holds. Therefore $P_n^{(p)} \circ \xi = f \circ \xi - q_0 \circ \xi$ is an extremal function for the right side of (22), and the value of the minimum is $m_n^{(p)}(a, b) = m_n^{(p)}(0, |\Delta|)$. The result follows in this case.

By the Pólya algorithm (see [6, pp. 65–66]),

$$\lim_{p\to\infty} P_n^{(p)}(t) = P_n^{(\infty)}(t)$$

uniformly on $[a, b]$. By the generalized Pólya algorithm (see [3]), as $p \to \infty$ a subsequence of the extremal functions $P_n^{(p)}(\xi(t))$ for the right side of (22) converges uniformly on $\bar{\Delta}$ to an extremal function for the case $p = \infty$. The function obtained in the limit necessarily has the form $P_n^{(\infty)}(\xi(t))$, and the value of the minimum is

$$\max_{t\in\bar{\Delta}} | P_n^{(\infty)}(\xi(t))| = \max_{t\in[a,b]} | P_n^{(\infty)}(t)| = m_n^{(\infty)}(a, b) = m_n^{(\infty)}(0, |\Delta|).$$

The theorem follows.

It is interesting to note that the Pólya theorem cited in the introduction has an extension from $p = \infty$ to $1 < p < \infty$.

4.4 Corollary. *Assume $1 < p < \infty$. Then for every $n = 1, 2, 3,\ldots,$*

$$m_n^{(\infty)}(0, |\Delta|) \leqslant \min \left\{ \left\| t^n + \sum_{j=0}^{r-1} \alpha_j t^j \right\|_{L^p(\Delta)} : \alpha_j \text{ complex} \right\}. \tag{23}$$

and the inequality is strict unless Δ is a single interval (i.e., $r = 1$), in which case equality holds.

Proof. The inequality (23) follows from (22). Since the extremal function for (22) is automatically unique for $1 < p < \infty$, equality can hold in (23) if and only if $P_n^{(p)}(\xi(t))$ is a polynomial. The latter occurs exactly when Δ reduces to a single interval.

References

1. N. Aronszajn and W. F. Donoghue, Jr., On exponential representations of analytic functions in the upper half-plane with positive imaginary part, *J. Anal. Math.* 5 (1956–57), 321–388.
2. L. de Branges and J. Rovnyak, "Square Summable Power Series," Holt, Rinehart and Winston, New York, 1966.
3. R. C. Buck, Linear spaces and approximation theory, "On Numerical Approximation," (R. E. Langer, Ed.). University of Wisconsin Press, Madison, 1959.
4. J. V. Ryff, Subordinate H^p functions, *Duke Math. J.* 33 (1966), 347–354.
5. H. S. Shapiro, "Topics in Approximation Theory," Lecture Notes in Mathematics, Vol. 187. Springer–Verlag, New York, 1971.
6. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," Macmillan, New York, 1963.